Non-Abelian Magnetic Monopoles in a Background of Gravitation with Fermions

by

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Abstract

The purpose of this paper is to study static solution configuration which describes the magnetic monopoles in a scenary where the gravitation is coupled with Higgs, Yang-Mills and fermions. We are looking for analysis of the energy functional and Bogomol'nyi equations. The Einstein equations now take into consideration the fermions' contribution for energy-momentum tensor. The interesting aspect here is to verify that the fermion field gives a contribution for non abelian magnetic field and for potential which minimise the energy functional.

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1 Introduction

The study of magnetic monopoles started with the Dirac's paper in 1948 [3] where for the first time the possibility of existence of this kind of particles in nature was analysed.

Some years later t'Hooft and Polyakov again found the magnetic monopoles but in a different context, the so called non-abelian monopoles had a scalar field as a source.

More recently Wali [1, 2] has studied the Bogomol'nyi equations and he again obtained the magnetic monopole type of solution. The minimization of the functional of energy gives us the conditions for analysis of Bogomol'nyi equations.

Finally Christiansen at all [5] have studied the Bogomol'nyi equations in Gauge theory and they verify conditions on functional of energy which is suitable for obtaining finite energy for a static configuration of field (gauge and scalar field) in $D = 2 + 1$ dimensions.

In our case the objective is to study the Einstein-gauge-field-Higgs-fermions system in $D = 3 + 1$ dimensions and, following of Christiansen and Wali, we wish to know the contribution for magnetic field and the potential that minimizes the energy functional when we include fermions in the model. Second we verify the possibility of the existence of non abelian magnetic monopoles in $D = 3 + 1$ dimension with fermion contributions.

The paper is written with the following outline:

1. First we consider the complete system Higgs-gauge-fermions and coupling between gravitation and scalar field. We have established equations of motion for systems with and without fermions.

2. We review some results of Wali at al [1, 2] for Bogomol'nyi equations and for energy functional when fermions and gravitation are not present.

3. Next we consider the contribution of fermions alone but without gravitation. There is no coupling between the scalar field and gravitation. We obtain here the magnetic

field with fermionic contributions and the potential function that minizes the energy functional.

4. We verify if there is a structure for the magentic monopole when we consider the covariant divergence associated with the gauge field.

5. Finally we return to the original problem and consider the lagrangean with all fields and coupling between gravitation and scalar field. All the equations are solved for this case, but there remain two problems that have no solutions.

   The first problem is about the new magnetic field and the potential function when we account for the fermion contributions and the background of gravitation.

   The questions is: what is the new magnetic field and potential function that minimizes the functional of energy if we put fermions and gravitation?

   Second, what is the meaning of the presence of spinors in the initial lagrangean? What is the physical interpretation for spinors here?

   In reality we have been speaking about "fermions". However in a classical problem, the correct description is "spinors". The question is what is the meaning of the spinors in our problem?

   Anyway we have a partial solution for this problem and it will appear elesewhere.

Let us start with a non Abelian Higgs-gauge-"fermions" and coupling between scalar field and gravitation in $3+1$ dimensions. We would like to verify the behaviour of magnetic monopole of t'Hooft-Polyakov type in a background of gravitation with presence of fermions.

We take the action given by Wali [1] [2] for a system of Einstein-Yang-Mills-Higgs in

the form:

$$S=\int d^4x\sqrt{-g}\ \left[\frac{1}{16\pi Gv^2}\ R\phi^2-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2+\frac{1}{2}\left(D_\mu\phi^a\right)^2-\frac{\lambda}{4}\left(\phi^2-v^2\right)^2\right] \tag{1.1}$$

and now we will consider the Dirac Lagrangean

$$\mathcal{L}=\overline{\psi}^i_\alpha\left(i\gamma^\mu_{\alpha\beta}\mathcal{D}_\mu-m\delta_{\alpha\beta}\right)\psi^i_\beta+\chi\overline{\psi}^i_\alpha\phi^a\left(T^a\right)_{ij}\psi^j_\alpha \tag{1.2}$$

for fermion contribution.

The matrix T^a are hermitian with null trace. It describes the three generators of $SU(2)$ group, here $a=1,2,3$ in the N-dimensional representation. The indices $i,\ j=1,\cdots N$, where N represent the dimension of given irreducible representation of $SU(2)$. The χ is only a constant.

The signature of metric is given by $(+---)$. The indices μ,ν take values from $\underline{0}$ to $\underline{3}$ and the indices $i,\ j$ assume values between 1 and 3. The indices a,b vary with the representation of a given gauge group.

We define R as the scalar curvature, ${F_{\mu\nu}}^a$ represents the stress field associated with the gauge field A^a_μ given as

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu \tag{1.3}$$

where f^{abc} represent the structure constants of the group. The gauge covariant derivative associated with Higgs field ϕ^a is given by

$$D_\mu\phi^a=\partial_\mu\phi^a+gf^{abc}A^b_\mu\phi^c\ . \tag{1.4}$$

A new covariant derivative associated with the spinor field is necessary here. The fermion field derivative ψ^i_β is given by

$$\mathcal{D}_\mu\psi^i_\beta=\partial_\mu\psi^i_\beta-igA^a_\mu\left(T^a\right)_{ij}\psi^j_\beta \tag{1.5}$$

where g is the coupling constant.

The term $\frac{\lambda}{4}\left(\phi^2 - v^2\right)^2$ represents the possibility of gauge symmetry breaking. We choose the unit system such that $4\pi G v^2 = 1$. Thus the action is written as

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\Big[-\frac{1}{4}R\phi^2 - \frac{1}{4}\left(F_{\mu\nu}{}^a\right) + \frac{1}{2}\left(D_\mu\phi^a\right)^2 + U\left(\phi_i\right) + \\ &\quad \overline{\psi}^i_\alpha\left(i\gamma^\mu_{\alpha\beta}\mathcal{D}_\mu - m\delta_{\alpha\beta}\right)\psi^i_\beta + \chi\overline{\psi}^i_\alpha\phi^a\left(T^a\right)_{ij}\psi^j_\alpha\Big] . \end{aligned} \tag{1.6}$$

Here we use $U\left(\phi_i\right)$; $i = 1, 2, 3$ in the form given by (1.1).

The equations of motions for Yang-Mills and Higgs fields when fermion are not considered are given respectively by

$$\frac{1}{\sqrt{-g}}D_\mu\left(\sqrt{-g}\;F^{\mu\nu a}\right) = -gf^{abc}\phi^b D^\nu\phi^c , \tag{1.7}$$

$$\frac{1}{\sqrt{-g}}D_\mu\left(\sqrt{-g}\;D^\mu\phi^a\right) = -\left[\frac{R}{2} + \lambda\left(\phi^2 - v^2\right)\right]\phi^a \tag{1.8}$$

and the Einstein equations are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{2}{\phi^2}T_{\mu\nu} \tag{1.9}$$

where the stress energy-momentum tensor for Einstein Yang-Mills-Higgs is written as

$$T_{\mu\nu} = g_{\mu\nu}T - F_{\mu\rho}{}^a F_\nu{}^{\rho a} + D_\mu\phi^a D_\nu\phi^a - \frac{1}{2}\phi^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \tag{1.10}$$

and

$$T = \frac{1}{4}F_{\mu\nu}{}^a F^{\mu\nu a} = \frac{1}{2}D_\mu\phi^a D_\nu\phi^a + \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 \tag{1.11}$$

The contribution to the energy-momentum due the fermions is given by

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2}i\left(\overline{\psi}^i_\alpha\gamma_{\nu\alpha\beta}\mathcal{D}_\mu\psi^i_\beta + \overline{\psi}^i_\alpha\gamma_{\mu\alpha\beta}\mathcal{D}_\nu\psi^i_\beta\right) + g_{\mu\nu}\left(m\overline{\psi}^i_\alpha\psi^i_\beta\delta^{\alpha\beta} + \right. \\ &- \left. ie^\chi_a\psi^i_\alpha\gamma^a_{\alpha\beta}\mathcal{D}_\chi\psi^i_\beta - \chi\overline{\psi}^i_\alpha\phi^a\left(T^a\right)_{ij}\psi^i_\alpha\right) \end{aligned} \tag{1.12}$$

Now with fermions, the equations of motion for fields A_μ^a ϕ^a and ψ_β^i are given respectively by

$$\partial_\beta F_{\alpha\beta}{}^d = gf^{acd}\Big[F_{\alpha\nu}{}^a A_\nu^c - \left(D_\alpha\phi^a\right)\phi^c\Big] + g\psi_\varepsilon^i\gamma_{\varepsilon\gamma}^\mu\delta_{\mu\alpha}\left(T^d\right)_{ij}\psi_\gamma^j \ ,$$
$$\left(-\frac{1}{\sqrt{-g}}\ \partial_\mu\sqrt{-g}\right)D_\mu\phi^a = D_\mu\left(D_\mu\phi^a\right) + \left[\frac{R}{2} + \lambda\left(\phi^2 - v^2\right)\right]\phi^a - \chi\overline{\psi}_\alpha^i\left(T^a\right)_{ij}\psi_\alpha^j \ , \tag{1.13}$$

$$-\frac{1}{\sqrt{-g}}\left(\partial_\mu\sqrt{-g}\right)\bar{\psi}_\alpha^k i\gamma_{\alpha\gamma}^\mu = \gamma_{\alpha\gamma}^\mu\mathcal{D}_\mu\bar{\psi}_\alpha^k + m\bar{\psi}_\alpha^i\delta_\alpha^\gamma\delta_k^i - \chi\bar{\psi}_\gamma^i\phi^a\left(T^a\right)_{ik} \ , \tag{1.14}$$

2 Static Equations and Bogomol'nyi Conditions

We wish to get only static solutions (time independent) for the system described by (1.6). Thus, using the technique of Bogomol'nyi, with appropriate boundery conditions [1, 2, 5],

$$D\phi = 0 \quad , \quad \phi^2 = v^2 \ . \tag{2.1}$$

The gauge field without fermions is given by [5]

$$A_i^a = \frac{1}{gv^2}\ \varepsilon^{abc}\phi^b\partial_i\phi^c + \frac{1}{v}\ \phi^a A_i \tag{2.2}$$

where A_i is arbitrary and $F_{ij}{}^a$ satisfy

$$\phi^a F_{ij}{}^a = \frac{\phi^2}{v}\ \mathcal{F}_{ij} \tag{2.3}$$

and the field $\mathcal{F}_{ij}$ is given by

$$\mathcal{F}_{ij} = \frac{1}{gv\phi^2}\ \varepsilon^{abc}\phi^a\partial_i\phi^b\partial_j\phi^c + \partial_i A_j - \partial_j A_i \ . \tag{2.4}$$

Only the static abelian gauge field will survive for long distances. Then, we define the "magnetic field" $\mathcal{B}_i$ associated with the monopole for long distance as

$$\mathcal{B}^i = \frac{1}{2}\ \varepsilon^{abc}\mathcal{F}_{jk} = \frac{1}{2gv^3}\ \varepsilon^{ijk}\varepsilon^{abc}\phi^a\partial_j\phi^b\partial_k\phi^c + \varepsilon^{ijk}\partial_j A_k \ . \tag{2.5}$$

The magnetic charge of the configuration is given by

$$g = \frac{1}{4\pi} \int d^3x \partial_i \mathcal{B}^i = \frac{1}{8\pi g v^3} \int_{s^2_\infty} d\sigma_i \varepsilon^{ijk} \varepsilon^{abc} \phi^a \partial_j \phi^b \partial_k \phi^c = \frac{n}{g} \tag{2.6}$$

where

$$n = \frac{1}{8\pi v^3} \int_{s^2_\infty} d\sigma_i \varepsilon^{ijk} \varepsilon^{abc} \phi^a \partial_j \phi^b \partial_k \phi^c \ . \tag{2.7}$$

which is a topological number.

Taking the limit $\frac{\lambda}{g} \rightarrow 0$, the functional of energy obtained from (1.6), without fermions for the flat spacetime is given by

$$\varepsilon = \int d^3x \left(\frac{1}{4} \ F_{ij}{}^a F^{ija} - \ \frac{1}{2} \ D_i \phi^a D^i \phi^a \right) \tag{2.8}$$

that is an energy of Higgs-Yang-Mills field.

For the case with the fermions, we have the energy functional written as,

$$\begin{aligned} \varepsilon \ &= \ \int d^3x \Big[\frac{1}{4} \ F_{ij}{}^a F^{ija} - \frac{1}{2} \ D_i \phi^a D^i \phi^a + \overline{\psi}^i_\alpha \left(m \delta_{\alpha\beta} - i \gamma^k_{\alpha\beta} \mathcal{D}_k \right) \psi^i_\beta + \\ &- \ \chi \overline{\psi}^i_\alpha \phi^a \left(T^a \right)_{ij} \psi^j_\alpha + U\left(\phi_i \right) \Big] \ . \end{aligned} \tag{2.9}$$

We are not considering the coupling between the scalar and gravitation here.

¿From eq. (2.8) we can define the electric and magnetic fields as:

$$E_i{}^a = D_i \phi^a \ , \tag{2.10}$$

$$B^a_i = \frac{1}{2} \ \varepsilon_{ijk} F^{jka} \tag{2.11}$$

for the case when the fermions are not present and when the flat spacetime is considered ($R = 0$); in other words, the eq. (2.10) and (2.11) are good definitions for the case when we don't consider the gravitation background. If we use the radiation gauge, $A^a_0 = 0$, and for the static solution $D_0 \phi^a = 0$, from eq. (1.3) and (1.4) it follows that

$$\begin{aligned} F_{oi}{}^a \ &= \ \partial_o A^a_i - \partial_i A^a_o + g f^{abc} A^b_o A^c_i \ , \\ F_{oi}{}^a &= 0 = D_i \phi^a \ . \end{aligned} \tag{2.12}$$

Then we have obtained a solution (null electricaly), because the electric field is zero in the whole space.

We have still [1, 2] the following inequality.

$$\begin{aligned}
\varepsilon &= \frac{1}{2}\int d^3x\left[(E_i^a)^2+(B_i^a)^2\right]^2 \\
&= \frac{1}{2}\int d^3x\,(E_i^a \mp B_i^a)\,(E_i^a \mp B_i^a) \pm \int d^3x E_i^a B_i^a \geq \pm \int d^3x E_i^a B_i^a \ . \qquad (2.13)
\end{aligned}$$

Using now the Bianchi indetity for ${F_{ij}}^a$, we can write

$$\pm \int d^3x E_i^a B_i^a = \pm \int d^3x \partial_i \left(\frac{1}{2}\,\varepsilon^{ijk} F_{jk}^a \phi^a\right) \qquad (2.14)$$

Comparing eq. (2.14) with eq. (2.3) the surface integral in eq. (2.14) asymptotically takes the value

$$\pm\, v \int d^3x \partial_i B^i = \frac{4\pi n v}{g} \ . \qquad (2.15)$$

On the other hand, if the Bogomol'nyi equations

$$E_i^a = B_i^a \qquad (2.16)$$

are satisfied, then the functional of energy is definitely minimized. It was shown in [1] [2], that this follows naturely from the relations (2.13) – (2.16).

Now, consider the case described by (2.9) where the fermions are present but without the background of gravitation.

We can still write suitably the energy functional (2.9) in terms of fields E_i^a and B_i^a with the same arguments. However, here it will not have the electric component due to the presence of the fermions. The magnetic field will be different since it shall have the contribution of fermions.

The scalars fields will be treated as a condensate of fermions. We shall define the

following quantities:

$$\begin{aligned}\eta &= m\overline{\psi}^i_\alpha \psi^i_\beta \delta_{\alpha\beta} \ , \\ \xi &= i\overline{\psi}^i_\alpha \gamma^k_{\alpha\beta} \ \mathcal{D}_k \psi^i_\beta \ , \\ \text{and} \qquad \Delta &= \chi \overline{\psi}_\alpha{}^i \phi^a \left(T^a\right)_{ij} \psi^j_\alpha \ .\end{aligned} \tag{2.17}$$

We have created three new scalar fields (η, ξ, Δ). So, our Lagrangean (1.6) with the scalar curvature $R = 0$ or its form eq. (2.8) with the fermions present, it is not possible anymore to obtain an energy functional that is satured by $\lambda\phi^4$ potential as in [5].

Using now the same prescription as in [5] it is conjectured that it is possible to verify that eq. (2.9) in $D = 3+1$ dimensions may be reduced to the following form:

$$\varepsilon = \frac{ev^2}{2}\ \Phi_{B^a_k} \pm \frac{1}{2e}\oint d\sigma_i J^a_j \varepsilon^{ijk}\varepsilon^{abc}\phi^b\partial_k\phi^c \mp \int d^3x \Big\{ \frac{1}{2}\left(B^a_k\partial_k\phi^a \mp \sqrt{2U}\right)^2$$
$$\mp \int \left[\mp\frac{e}{2}\left(v^2 - |\phi|^2\right)\left(\eta + \xi + \Delta\right)\left(\partial_k\phi^a\right) \mp \sqrt{2U} \right] B^a_k \partial_k\phi^a + \frac{1}{2}\left|\left(D_1 \pm iD_2\right)\phi^a\right|^2 \Big\} . \tag{2.18}$$

Here $\Phi_{B^a_k}$ means the non-abelian magnetic flux. The second term is a surface term and goes to zero at infinity since all fields go to zero at infinity.

We need to discover what magnetic field B^a_k and potential $U(\phi_i)$ will minimize the energy functional or in other words, what are B^a_k and $U(\phi_i)$ which will saturate the functional, ε?

If we use the duality condition [5] as in the form

$$D_1\phi^a = \mp iD_2\phi^a \tag{2.19}$$

and noting that we are considering only configurations with finite energy the surface integral of current vector is null.

In the Bogolomol'nyi limit given by eq. (2.18) the minimum of energy is obtained exactly if

$$\varepsilon = \frac{ev^2}{2}\ \Phi_{B^a_k} \ . \tag{2.20}$$

Since we wish to saturate the functional ε the magnetic field now carrying the fermion's contribution will be given by

$$\mathcal{B}_k^a = \frac{e}{2}\left(v^2 - |\phi|^2\right)(\eta + \xi + \delta)\left(\partial_k \phi^a\right) \ . \tag{2.21}$$

and the new potential will be written as

$$U\left(\phi^a, \eta, \xi, \Delta\right) = \frac{e}{8}\left(|\phi|^2 - v^2\right)^2 (\eta + \xi + \Delta)^2 \left(\partial_k \phi^a\right)^4 \tag{2.22}$$

Clearly the potential is now of type $\lambda\phi^8$ because of fermion contribution.

With the definition of the covariant derivative given by

$$D_\mu = \partial_\mu + i\tilde{g}A_\mu$$

we get

$$D_\mu F^{\mu\nu a} = \partial_\mu F^{\mu\nu a} + i\tilde{g}\left[A_\mu, F^{\mu\nu}\right]^a \ .$$

Thus, the covariant divergence is given as

$$D_\mu F^{\mu\nu a} = \vec{\nabla} \cdot B^a + \tilde{g} f^{abc} \vec{A}_b \cdot \vec{B}_c = 0 \tag{2.23}$$

or still in a compact form,

$$D_\mu F^{\mu\nu a} = \vec{D} \cdot \vec{B} = 0 \tag{2.24}$$

where

$$D = \vec{\nabla} + \tilde{g} f^{abc} \vec{A} \ . \tag{2.25}$$

Since B_k^a is given by eq. (2.21) and $|\phi|^2$ is written as

$$|\phi|^2 = \phi^a \phi_a \tag{2.26}$$

a suitably ansatz for scalar field may be chosen such as

$$\phi^a = F \ \frac{r^a}{r} \quad (r \to \infty) \ . \tag{2.27}$$

Using eq. (2.27) and eq. (2.26) in equation eq. (2.23) or eq. (2.24) above we get

$$\vec{\nabla} \cdot \vec{B}_k^a \sim \frac{g}{r^2} \ . \tag{2.28}$$

This gives us a structure of the magnetic monopole where g represents the source of the magnetic field (non-Abelian magnetic field) whose source has the origin in the scalar field, gauge field and condensate of fermions $\left(\phi^a, A_\mu^a, \eta, \xi, \Delta\right)$. It's sufficient to choose or to fix the "F" function such as that form to obtain the Gauss law from eq. (2.28).

The energy functional in a curved spacetime

Now the same problem is proposed in a curved spacetime.

The energy functional (static functional) that is obtained from eq. (1.6) in a curved spacetime is described by.

$$\begin{aligned}\varepsilon \;\; = \;\; & \int d^3x\sqrt{-g}\Big[\frac{1}{4}\ R\phi^2 + \frac{1}{4}\ F_{ij}{}^a F^{ija} - \frac{1}{2}\ D_i\phi^a D^i\phi^a + \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 + \\ & \overline{\psi}_\alpha^i\left(m\delta_{\alpha\beta} - i\gamma_{\alpha\beta}^k \mathcal{D}_k\right)\psi_\beta^i - \chi\overline{\psi}_\alpha^i\phi^a\left(T^a\right)_{ij}\psi_\alpha^j\Big] \ .\end{aligned} \tag{2.29}$$

The energy functional can, in principle, be minimized with the gravitation as a background field only if we introduce a third covariante derivative associated with gravitation, but we need also to transfer the dynamics from the metric to vierbein and to write an appropriate covariante derivative carrying the spin connection such as:

$$\tilde{D}_\mu = \partial_\mu + \frac{1}{8}\ i\left[\gamma_a\gamma_b\right]B_\mu^{ab}$$

where

$$\gamma^\mu = e_a^\mu(x)\gamma^a$$

and

$$g_{\mu\nu} = e_\mu^a e_\mu^b \eta_{ab}$$

are Dirac's matrices and metric respectively written in a local Lorentz coordinate system, e^a_μ are the vierbeins and $B^{ab}_\mu(x)$ are spin connections, completely determined by vierbeins. Now the gravitation is considered under flat space-time given by η_{ab} in Minkowski space.

3 The basic Equations

We choose the spherically symmetric static metric as

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - C^2(r)r^2 d\Omega^2 \ . \tag{3.1}$$

The Einstein tensor and the scalar curvature are easily obtained,

$$\begin{aligned}
G_{00} &= \frac{A^2}{B^2}\left[\frac{1}{r^2}\left(\frac{B^2}{C^2}-1\right)+\frac{2}{r}\,\frac{B'}{B}+2\,\frac{B'}{B}\,\frac{C'}{C}-\frac{2C''}{C}-\frac{6C'}{rC}-\left(\frac{C'}{C}\right)\right] \ , \\
G_{rr} &= \left(\frac{C'}{C}\right)^2+\frac{2}{r}\,\frac{C'}{C}+\frac{1}{r^2}\left(1-\frac{B^2}{C^2}\right)+\frac{2C'}{C}\,\frac{A'}{A}+\frac{2}{r}\frac{A'}{A} \ , \\
G_{\theta\theta} &= \frac{C^2r^2}{B^2}\left(\frac{A''}{A}+\frac{1}{r}\,\frac{A'}{A}-\frac{A'}{A}\,\frac{B'}{B}+\frac{C'}{C}\,\frac{A'}{A}-\frac{1}{r}\,\frac{B'}{B}-\frac{B'}{B}\,\frac{C'}{C}+\frac{C''}{C}+\frac{2}{r}\,\frac{C'}{C}\right) \ , \\
G_{\phi\phi} &= sen^2\theta G_{\theta\theta} \ ,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
R &= \frac{2}{r^2}\left(\frac{1}{B^2}-\frac{1}{C^2}\right)+\frac{2}{B^2}\left[\frac{A''}{A}+\frac{2}{r}\,\frac{A'}{A}-\frac{B'}{B}\,\frac{A'}{A}-\frac{2}{r}\,\frac{B'}{B}-\frac{2B'}{B}\,\frac{C'}{C}+\right. \\
&\left.\frac{2C'}{C}\,\frac{A'}{A}+\frac{2C''}{C}+\frac{6C'}{rC}+\frac{C'}{C}\right]
\end{aligned} \tag{3.3}$$

Then if we consider the eq. (1.10) and (1.11) together with the eq. (3.2) we take components of the energy momentum tensor for the global system of Einstein-gauge-Higgs-fermions.

The components are given by:

$$
\begin{aligned}
\tilde{T}_{00} &= \frac{\phi^2 A^2}{B^2}\left[\frac{2C'}{C}\;\frac{A'}{A}+\frac{2}{r}\;\frac{A'}{A}-\frac{A'}{A}\;\frac{B'}{B}+\frac{A''}{A}-\frac{1}{2}\left(\frac{A'}{A}\right)^2+\frac{\lambda v^2}{4}\;\frac{\left(h^2-1\right)^2}{h^2}+\right.\\
&\quad\left.\frac{2h''}{h}+\left(\frac{2h'}{h}\right)+\frac{2h'}{h}\left(\frac{2C'}{C}+\frac{2}{r}-\frac{B'}{B}\right)+\frac{3A'}{A}\;\frac{h'}{h}\right]+\tau_{00}\ ,\\
\tilde{T}_{rr} &= \phi^2\left[-\frac{1}{2}\left(\frac{A'}{A}\right)^2-\frac{A'}{A}\;\frac{h'}{h}+\frac{2B'}{B}\;\frac{h'}{h}-\frac{\lambda v^2}{4}\;B^2\frac{\left(h^2-1\right)^2}{h^2}\right]+\tau_{11}\ ,\\
\tilde{T}_{\theta\theta} &= \frac{C^2 r^2}{B^2}\;\phi^2\left[\frac{1}{2}\left(\frac{A'}{A}\right)^2-\frac{h''}{h}-\left(\frac{h'}{h}\right)^2-\frac{h'}{h}\left(\frac{2C'}{C}+\frac{2}{r}-\frac{B'}{B}\right)+\right.\\
&\quad\left.-\frac{\lambda v^2}{4}\;\frac{B^2\left(h^2-1\right)^2}{h^2}\right]+\tau_{22}\ ,\\
\tilde{T}_{\phi\phi} &= sen^2\theta\tilde{T}_{\theta\theta}+\tau_{33}
\end{aligned}
\tag{3.4}
$$

where $h = h(r) = \dfrac{\phi^a}{vr^a}$ and $\tau_{00},\ \tau_{11},\ \tau_{22},\ \tau_{33}$ are the components of the energy-momentum tensor for fermions given by

$$
\begin{aligned}
\tau_{00} &= i\overline{\psi}^i_\alpha\gamma_{0\alpha\beta}\mathcal{D}_0\psi^i_\beta+g_{00}\left(m\overline{\psi}^i_\alpha\psi^i_\beta\delta^\alpha_\beta-ie^\chi_a\psi^i_\alpha\gamma^a_{\alpha\beta}\mathcal{D}_\chi\psi^i_\beta-\chi\psi^i_\alpha\phi^a\left(T^a\right)_{ij}\psi^j_\alpha\right)\ ,\\
\tau_{11} &= i\overline{\psi}^i_\alpha\gamma_{1\alpha\beta}\mathcal{D}_1\psi^i_\beta+g_{11}\left(m\overline{\psi}^i_\alpha\psi^i_\beta\delta^\alpha_\beta-ie^\chi_a\psi^i_\alpha\gamma^a_{\alpha\beta}\mathcal{D}_\chi\psi^i_\beta-\chi\overline{\psi}^i_\alpha\phi^a\left(T^a\right)_{ij}\psi^j_\alpha\right)\ ,\\
\tau_{22} &= i\overline{\psi}^i_\alpha\gamma_{2\alpha\beta}\mathcal{D}_2\psi^i_\beta+g_{22}\left(m\overline{\psi}^i_\alpha\psi^i_\beta\delta^\alpha_\beta-ie^\chi_a\psi^i_\alpha\gamma^a_{\alpha\beta}\mathcal{D}_\chi\psi^i_\beta-\chi\overline{\psi}^i_\alpha\phi^a\left(T^a\right)_{ij}\psi^j_\alpha\right)\ ,\\
\tau_{33} &= i\overline{\psi}^i_\alpha\gamma_{3\alpha\beta}\mathcal{D}_3\psi^i_\beta+g_{33}\left(m\overline{\psi}^i_\alpha\psi^i_\beta\delta^\alpha_\beta-ie^\chi_a\psi^i_\alpha\gamma^a_{\alpha\beta}\mathcal{D}_\chi\psi^i_\beta-\chi\overline{\psi}^i_\alpha\phi^a\left(T^a\right)_{ij}\psi^j_\alpha\right)
\end{aligned}
\tag{3.5}
$$

where e^χ_a imply the tetrads or vierbeins.

This set of equations together with the Bogomol'nyi equations are the basic equations for our system, Einstein-gauge-Higgs-fermions.

Now the next step is to find solutions for the Einstein equations with the objective to find the monopoles appearing in this gravitational background when the fermions are present. But on the other hand when we have interaction between spinors and gravitation field the unique way to consider that coupling in our case is only if we introduce a local Lorentz coordinate system. Thus, we do not have curved space-time anymore.

Conclusions

We have analysed two types of systems. One of them with Yang-Mills and scalar fields in flat spacetime and other consisting of the Yang-Mills-scalar field and "fermions".

We treated "the fermions" as a condensate of scalars fields. In this case the magnetic field that saturated the energy functional has contributions from fermions. The potential which minimized the same functional is of the kind $\lambda\phi^8$ and thus different from [5] whose potential is of type $\lambda\phi^4$ for $2+1$ dimension case.

For the case of Yang-Mills-scalar field [1, 2] the Bogomol'nyi equations have a simple solution in flat space time. In the present case our conjecture in eq. (2.18) gives us the non abelian magnetic field much more complicated and potential with fermion's contribution. A structure of magnetic monopoles can be found with a field given by eq. (2.28).

Finally we have assumed the system Einstein-Higgs-scalar field-gauge and we have obtained all the equations for this system but solutions to the Einstein equations are lacking. Magnetic monopoles appear in a new context and it's still necessary to find in a new magnetic field B_k^a and potential $U(\phi_i)$ that will be suitable for minimising the functional of energy when the background gravitational field is considered. At this point we have a conflict between a curve space-time and a local lorentz manifold. On the first way we have the the complete set of equations plus Bogolmo'ni equations being the basic equations for Einstein-gauge-Higgs-fermions system- and, in principle, the energy functional could be minimized with the graviation field as a background, but the other hand because the spinors we need to introduce a local coordinates system and to proceed the minimization of the functional of energy via vierbeins and spin connection. This will be a part of a separate analysis.

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